

Exact time-dependence of causal correlations and non-equilibrium density matrices in holographic systems

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We present the first exact calculations of time-dependence of causal correlations in driven non-equilibrium states in $(2+1)$ -D systems using holography. Comparing exact results with those obtained from simple prototype geometries that are parametrised only by a time-dependent temperature, we find that the universal slowly-varying features are controlled just by the pump duration and the initial and final temperatures. The exact locations of the event and apparent horizons in the dual geometries can be deduced from the causal correlations without any prior knowledge of the dual gravity theory.

Introduction: Hydrodynamics gives us a general understanding of how expectation values of local operators *e.g.* the energy-momentum tensor and conserved currents in many-body systems thermalise. A similar general understanding of non-equilibrium time evolution of correlation functions is still elusive. This is particularly hard for the case of the out-of-time-ordered (i.e. unequal time) correlation functions in which case non-perturbative techniques are essential even at weak coupling [1–3]. The chief objective of this work is to perform the first *exact* calculations of the time-dependence of the causal (a.k.a. retarded) correlations in a $(2+1)$ -D many-body system in states transitioning from an initial thermal equilibrium to another driven by a *homogeneous* energy injection from a source (a.k.a. pump). To that end we apply the holographic correspondence [4] that maps strongly interacting many body systems to classical theories of gravity with a few dynamical fields in one higher dimension. Furthermore, since the time-dependent causal correlation function can be known experimentally via techniques such as solid-state pump-probe spectroscopy [5], it is desirable to establish a general theory of its thermalisation.

Our computations reveal that at least in the regimes where the time duration of energy injection t_p is small compared to the initial thermal scattering time T_{in}^{-1} , the time-dependence of causal correlation functions in holographic systems has well-defined universal features. Namely, these features can be reproduced with better than $\mathcal{O}(t_p T_{\text{in}})$ accuracy by a simple prototype gravitational geometry that can be constructed using (i) the experimentally controlled t_p and (ii) the initial and final temperatures as the *only* inputs. Thus, the universal features of thermalisation can be understood without detailed knowledge of microscopic dynamics whose only role here is to determine the final temperature of the system. This indicates that many features of thermalisation of the causal correlations are controlled via simple parameters analogous to the Reynolds number for hydrodynamic flows.

This result is tied to the second motivation of our work which is to get a deeper understanding of the holographic duality itself. One of the fundamental questions is, if a large N quantum system is holographic, can the dual classical gravity theory be constructed directly from the observables themselves. We can rephrase the question in this form: having measured the time-dependent expectation values of local operators and their correlation functions in

a given non-equilibrium state, can we construct the dual gravitational geometry or at least know some of its defining features *without* prior knowledge of the dual classical gravity equations?

We will demonstrate here that the universal features of the time-dependent causal correlations reveal the *exact* location of the event horizon in the dual geometry. Furthermore, if we combine both universal and non-universal features of time-dependent causal correlations, we can extract the *exact* location of the apparent horizon in an appropriate bulk coordinate system. Thus *without* prior knowledge of the dual classical gravity equations, we can extract the exact locations of the event and apparent horizons of the dual geometries from measurements of causal correlations.

The procedure of extracting the exact locations of the event and apparent horizons from the causal correlations will be achieved via construction of prototype geometries that are dual to instantaneous thermal density matrices with appropriate time-dependent temperatures $T(t)$. We will see indications that we can systematically unravel the enormous complexity of non-equilibrium states via simple density matrix approximations.

Driven non-equilibrium holographic states: We consider a generic non-equilibrium state in a $(2+1)$ -D large- N conformal field theory (CFT) driven by the Hamiltonian $H(t) = H_{\text{CFT}} + H_{\text{pump}}(t)$ from one thermal equilibrium to another with initial and final temperatures T_{in} and T_{f} respectively. Here, $H_{\text{pump}}(t)$ represents energy injection from a pump – a *homogeneous* external source $f(t)$ coupling to a scalar operator O of the CFT, i.e. $H_{\text{pump}}(t) = \int d^2\mathbf{x} f(t) O(\mathbf{x})$. For our specific construction, O is assumed to have scaling dimension $\Delta = 2$ (like the electronic density operator at weak coupling). The external source $f(t)$ (which has the dimension of energy) is assumed to have a Gaussian profile:

$$f(t) = E_{\text{max}} e^{-t^2/2\sigma^2}. \quad (1)$$

The effective duration of the pumping is thus $|t| < t_p/2$ with $t_p \sim 6\sigma$. The final temperature T_{f} will be determined by the microscopic dynamics as functions of T_{in} , E_{max} and t_p . Since, the underlying microscopic theory is conformal, we choose units of measurement where $T_{\text{in}} = 1$. We focus on the case $E_{\text{max}} \sim T_{\text{in}}$ and $t_p T_{\text{in}} \ll 1$ arguing later why our results will be independent of the specific choice of $f(t)$ as long as these conditions are satisfied.

Holographically, such a driven state is represented by a $(3+1)$ -D asymptotically anti-de Sitter (AdS) metric G_{MN} , and a scalar field Φ (dual to the operator O) with mass given by $m^2 = -2/l^2$ and minimally coupled to Einstein gravity with cosmological constant $\Lambda = -3/l^2$. It is convenient to choose coordinates where G_{MN} and Φ take the form

$$ds^2 = \frac{l^2}{r^2} (-2drdv - A(r, v)dv^2) + S^2(r, v)(dx^2 + dy^2),$$

$$\Phi = \Phi(r, v), \quad (2)$$

where we have imposed homogeneity in the field-theory spatial coordinates x and y . At the boundary $r = 0$, the bulk coordinate v is identified with the field-theory time-coordinate t .

To achieve a unique gravitational solution, we need to provide (i) initial conditions for $S(r)$, $\Phi(r)$ and $\lim_{r \rightarrow 0} (1/6)\partial_r^3 A(r) = a_{3\text{in}}$ at $v = v_{\text{in}}$ in the far past, and (ii) the boundary conditions $\lim_{r \rightarrow 0} A(r, v) = a_0(v)$, $\lim_{r \rightarrow 0} rS(r, v) = s_0(v)$ and $\lim_{r \rightarrow 0} r^{-1}\Phi(r, v) = \Phi_0(v)$. By the holographic dictionary $a_0(v) = s_0(v) = 1$, since the dual system lives on flat Minkowski metric, and $\Phi_0(v)$ is identified with the source of the dual operator $f(v)$. Our initial state is thermal, therefore the initial conditions are set via a black hole geometry with mass M_{in} so that $S(r, v_{\text{in}}) = 1/r$, $a_{3\text{in}} = -M_{\text{in}} = -(4\pi/3)l^2 T_{\text{in}}^3$ and $\Phi(r, v_{\text{in}}) = 0$. The gravitational solution is obtained numerically via the method of characteristics [6, 7] as described in the Supplemental Material. From this solution, we can extract the energy density $\langle t_{00}(t) \rangle$, the pressure $\langle t_{xx}(t) \rangle = \langle t_{yy}(t) \rangle$ and the expectation value $\langle O(t) \rangle$ in the dual driven state via the holographic renormalisation procedure [8].

Time-dependent causal correlations: Linear response theory tells us that if we perturb the time-evolution of the non-equilibrium state driven by the pump via a probe perturbation $\Delta H = \gamma \int d^2\mathbf{x} \tilde{f}(t, \mathbf{x}) \tilde{O}(\mathbf{x})$, where \tilde{O} is an operator which is the same as or different from O to which the pump couples, then the time-dependent expectation value of $\langle \tilde{O}(\mathbf{k}) \rangle(t)$ is given by:

$$\delta \langle \tilde{O}(\mathbf{k}) \rangle(t) = \gamma \int dt' G_R^{\tilde{O}\tilde{O}}(t, t', \mathbf{k}) \tilde{f}(t', \mathbf{k}) + \mathcal{O}(\gamma^2). \quad (3)$$

Above $G_R^{\tilde{O}\tilde{O}}(t, t', \mathbf{k})$ is the causal correlation function (we have used the spatial homogeneity of the pump to Fourier transform the spatial $\mathbf{x} - \mathbf{x}'$ dependence). In the Heisenberg picture G_R takes the form:

$$G_R^{\tilde{O}\tilde{O}}(t, t', \mathbf{k}) = -i\theta(t - t') \text{Tr}(\rho_{\text{in}}[\tilde{O}(t, \mathbf{k}), \tilde{O}(t', \mathbf{k})]) \quad (4)$$

with ρ_{in} being the initial thermal density matrix at temperature T_{in} . Note that the time-evolution operator $U(t, t') = T \exp(-i \int dt' (H_{\text{CFT}} + H_{\text{pump}}(t')) dt')$ implicit in the definition above includes the pump, and therefore $G_R^{\tilde{O}\tilde{O}}(t, t', \mathbf{k})$ is not simply a function of $t - t'$ except in the far past and future when the pumping ceases and the states thermalise. Here, we will consider all possible cases in which the probe source $\tilde{f}(t)$ is turned on before, during or after the pump. For the sake of simplicity, we will consider \tilde{O} to be an operator of scaling dimension $\Delta = 2$ too.

In [9] (see [10–14] for earlier related works), a holographic prescription has been developed for obtaining

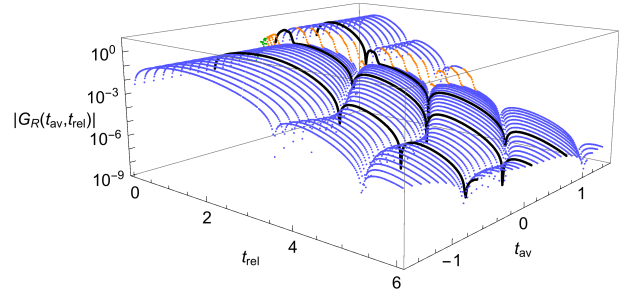


FIG. 1. The numerical result for G_R as a function of t_{av} and t_{rel} . Black curves indicate $G_R(t_{\text{rel}})$ for constant representative values of $t_{\text{av}} = -0.6, 0.2$ and 0.8 which are displayed in more detail in Fig. 2. See text for the color coding.

$G_R^{\tilde{O}\tilde{O}}(t, t', \mathbf{k})$ via a simple implementation of the linear response protocol described above in which we need to study the linearised fluctuation of the field $\delta\tilde{\Phi}(r, v, \mathbf{k})$, dual to $\tilde{O}(t, \mathbf{k})$, about the gravitational solution (2) representing the dual non-equilibrium state. The initial condition for $\delta\tilde{\Phi}(r)$ at $v = v_{\text{in}}$ is trivial by causality (since neither the pump nor the probe has been switched on) and the boundary condition is set by identifying the leading asymptotic $r \rightarrow 0$ mode with the probe source $\tilde{f}(t, \mathbf{k})$ i.e. imposing $\lim_{r \rightarrow 0} \delta\tilde{\Phi}(r, v, \mathbf{k})/r = \tilde{f}(v, \mathbf{k})$ for all times v . The solution for $\delta\tilde{\Phi}(r, v, \mathbf{k})$ is unique and can be obtained numerically via the method of characteristics [6, 7]. Further, we obtain the expectation value of the dual operator $\delta \langle \tilde{O}(t, \mathbf{k}) \rangle$ from this solution via holographic renormalisation [8]. Finally $G_R^{\tilde{O}\tilde{O}}(t, t', \mathbf{k})$ is extracted using the relation (3) with $\tilde{f}(v, \mathbf{k})$ chosen to be a narrow Gaussian profile appropriately normalized so that it can be treated as $\delta(v - t')$ up to any required order of numerical accuracy [9]. The latter feature then implies that $\delta \langle \tilde{O}(\mathbf{k}) \rangle(t) = G_R^{\tilde{O}\tilde{O}}(t, t', \mathbf{k})$. More details are presented in the Supplemental Material. The above prescription reproduces the results obtained with the well-known Son-Starinets prescription [15] for the thermal retarded correlation [9].

Exact results: The exact results for $G_R^{\tilde{O}\tilde{O}}(t, t', \mathbf{k})$ in the driven non-equilibrium state represented by the numerical gravitational solution (2) are presented in Fig. 1. We have set $\mathbf{k} = 0$ for presentation since we find (as in [9]) that our conclusions remain similar for $|\mathbf{k}| < T_{\text{in}}$. For better understanding, instead of using the probe time t' and observation time t for the plot, we have used the average time $t_{\text{av}} = (t + t')/2$ and the relative time $t_{\text{rel}} = t - t'$. We have also subtracted a (state-independent) contact term from $G_R^{\tilde{O}\tilde{O}}(t_{\text{av}}, t_{\text{rel}})$ which is localised at $t_{\text{rel}} = 0$. For the non-equilibrium geometry, we have chosen $\sigma = 0.03$ and $E_{\text{max}} = T_{\text{in}} = 1$ in Eq. (1), which leads to the final temperature $T_f \approx 2^{4/3}$. Clearly for large values of $|t_{\text{av}}|$, $G_R^{\tilde{O}\tilde{O}}(t_{\text{av}}, t_{\text{rel}})$ is independent of t_{av} and reduces to the initial/final thermal forms. Furthermore, since the underlying dynamics is conformal, at thermal equilibrium $G_R^{\tilde{O}\tilde{O}}(t_{\text{av}}, t_{\text{rel}}) = T^2 g_R^{\text{eq}}(t_{\text{rel}} T)$ (recall that $\Delta = 2$ for \tilde{O}). This is reflected in Fig. 1 through the appropriate t_{av} -evolution in the width, slope and height of the *ringdown* pattern of $G_R^{\tilde{O}\tilde{O}}(t_{\text{av}}, t_{\text{rel}})$ from its initial to final thermal forms.

Remarkably, this t_{av} -evolution of $G_R^{\tilde{O}\tilde{O}}(t_{\text{av}}, t_{\text{rel}})$ takes place over $-1.5 < t_{\text{av}} < 1.5$ if we impose 10^{-4} times the respective maximal thermal values cut-off on the maximum departures from the initial/final thermal forms. This overall time-scale (≈ 3) is about 100 times larger than

the root-mean-square-width $\sigma = 0.03$ of the pump and the time-scale of evolution of one-point functions with the same cut-off. It is however commensurate with the time-scale of the evolution of the location of the event horizon as we will see later. This is expected because the geodesic distance of the event horizon from the boundary controls the rate of dissipation, i.e. the effective quasi-normal mode pole which can be made precise via wavelet analysis [16] but will not be attempted here.

We divide the $t_{\text{av}} - t_{\text{rel}}$ plane into a *universal* and a non-universal region. The universal \mathcal{U} region (coloured in blue in Fig. 1) is defined as the region where the probe time $t' = t_{\text{av}} - t_{\text{rel}}/2$ is away from the pumping duration, i.e. $\mathcal{U} := \{t_{\text{rel}} < 2(t_{\text{av}} - 3\sigma)\} \cup \{t_{\text{rel}} > 2(t_{\text{av}} + 3\sigma)\}$. It is expected that the features of $G_R(t_{\text{av}}, t_{\text{rel}})$ in this region can be attributed mostly to the change of the location of the event horizon which will turn out to be universal, i.e. *independent of the details of the pumping protocol*, and determined largely by the initial and final temperatures, and the pumping duration only as we will see below. Therefore, these features are independent of the details of the gravity theory and hence the microscopic dynamics.

The non-universal region is further divided into two subregions. The first is the *probe-on-pump* (PP) region (coloured in orange in Fig. 1), which is defined as where the probe time $t' = t_{\text{av}} - t_{\text{rel}}/2$ is *within* but the observation time $t = t_{\text{av}} + t_{\text{rel}}/2$ is *away from* the pumping time, i.e. $|t'| < 3\sigma$ and $|t| > 3\sigma$. The features of $G_R(t_{\text{av}}, t_{\text{rel}})$ in this region (the most prominent one appears as an extra bump on the ringdown pattern) are not independent of the details of the microscopic dynamics. We will see below that these can be attributed nevertheless to the location of the apparent horizon (which responds to the pump instantaneously) and can be reproduced by a simple density matrix approximation dual to a simple prototype geometry. Conversely, the features of $G_R(t_{\text{av}}, t_{\text{rel}})$ in the \mathcal{U} and PP regions will allow us to *locate the exact event and apparent horizons of the dual geometry* (2) respectively, thus, helping us to deduce the dual gravity theory (i.e. the microscopic dynamics) as described below. The second (very tiny) subregion is the *probe-and-observation-on-pump* (POP) region (coloured in green in Fig. 1) where *both* the probe and observer time are within the pumping time (i.e. $|t|, |t'| < 3\sigma$). The features here cannot be reproduced by our simple approximations and hence will not be analysed here.

Approximations dual to simple density matrices:

The trial non-equilibrium geometries are AdS-Vaidya (AdSV) geometries defined only by a time-dependent black hole mass function $M(v)$ such that the functions $A(r, v)$ and $S(r, v)$ in (2) take the radically simple forms:

$$A(r, v) = 1 - M(v)r^3, \quad S(r, v) = 1/r, \quad (5)$$

while the scalar field Φ vanishes. These AdSV geometries are not viewed here as solutions of the dual classical gravity theory but rather as approximations representing instantaneously thermal density matrices $\rho_{\text{inst}}[T(t)] := \exp(-H_{\text{CFT}}/T(t))/\text{Tr}(\exp(-H_{\text{CFT}}/T(t)))$ which are not solutions of the microscopic time-evolution equations either. For this correspondence, we need to define $T(t)$ via $M(t) = (4\pi/3)l^2 T^3(t)$ as the instantaneous Hawking temperature of the AdSV geometry.

The simplest choice of $M(v)$ can be defined as a mono-

tonic interpolation between the initial and final black hole masses (M_{in} and M_{f} respectively) that is readily provided by a tanh-function whose time-scale of variation is the same as the root-mean-square-width σ of the pump, i.e.

$$M(v) = M_{\text{in}} + ((M_{\text{f}} - M_{\text{in}})/2)(1 + \tanh(v/\sigma)). \quad (6)$$

The above choice defines the $\text{AdSV}_{\mathcal{T}}$ prototype geometry examined in [9]. Crucially this prototype geometric approximation is parametrised *just by the initial and final temperatures and the pump duration*, and thus can be constructed without prior knowledge of the dual gravity theory.

The other prototype geometries important for the present discussion are $\text{AdSV}_{\mathcal{A}}$ and $\text{AdSV}_{\mathcal{E}}$ which can be constructed such that they reproduce the exact locations $r_{\text{EH}}(v)$ and $r_{\text{AH}}(v)$ of the event and apparent horizons of the numerical geometry respectively (see Supplemental Material for details and explanation why the constructions can be achieved avoiding ambiguities arising from diffeomorphism symmetries). Finally, we also consider $\text{AdSV}_{\mathcal{P}}$ as suggested in [17] where $M(v)$ is designed to reproduce $a_3(v) = \lim_{r \rightarrow 0} (1/6)\partial_r^3 A(r, v)$ of the numerical geometry exactly. Holographic renormalisation tells us that $P(v) = \langle t_{xx}(v) \rangle = \langle t_{yy}(v) \rangle = -a_3(v)$ is the pressure, therefore $\text{AdSV}_{\mathcal{P}}$ reproduces the exact time-evolution of the pressure by construction.

The comparisons of the time-dependent retarded correlation of the prototype geometries with that of the exact one are presented in Fig. 2 where we have chosen three representative values of t_{av} which are -0.6 , 0.2 and 0.8 . The insets for $t_{\text{av}} = 0.2$ and 0.8 show the PP regions on a linear scale.

Universal features, effective description and the horizons:

We readily see from Fig. 2 that in the \mathcal{U} region, all AdSV geometries approximate the exact retarded propagator within 1 percent relative accuracy which is much better than the naively expected $\mathcal{O}(3\sigma T_{\text{in}})$ i.e. 10 percent accuracy. A detailed analysis of the standard deviation (see Supplemental Material for details) reveals that the best approximation in the \mathcal{U} region is provided by $\text{AdSV}_{\mathcal{E}}$ prototype that reproduces the exact location of the event horizon. Remarkably, the approximations of $G_R(t_{\text{av}}, t_{\text{rel}})$ provided by the prototype geometries in the universal \mathcal{U} region can be intuitively explained by the fact that all these prototype geometries including $\text{AdSV}_{\mathcal{T}}$, $\text{AdSV}_{\mathcal{A}}$ and $\text{AdSV}_{\mathcal{P}}$ also reproduce the exact locations of the time-dependent event horizon within 1 percent relative accuracy for all times as shown in Fig. 3. Particularly the $\text{AdSV}_{\mathcal{T}}$ geometry which can be constructed without any detailed knowledge of the dual gravity theory reproduces the features of $G_R(t_{\text{av}}, t_{\text{rel}})$ in the \mathcal{U} region and also the location of the event horizon with remarkable high accuracy even *within* the pumping duration (see the inset plot in Fig. 3). So we can lay claim to their universality as mentioned before. The $\text{AdSV}_{\mathcal{T}}$ geometries have the feature that the location of the event horizon is determined mostly by the difference of the final and initial temperatures ΔT and is independent of the pumping duration (6σ) provided $3\sigma\Delta T \ll 1$ [18]. This can be arranged in the numerical geometry if $E_{\text{max}} \geq T_{\text{in}}$ and $\sigma T_{\text{in}} \ll 1$, and therefore under these conditions the features in the \mathcal{U} region should also be universal.

Conversely, since $\text{AdSV}_{\mathcal{E}}$ gives the best approximation

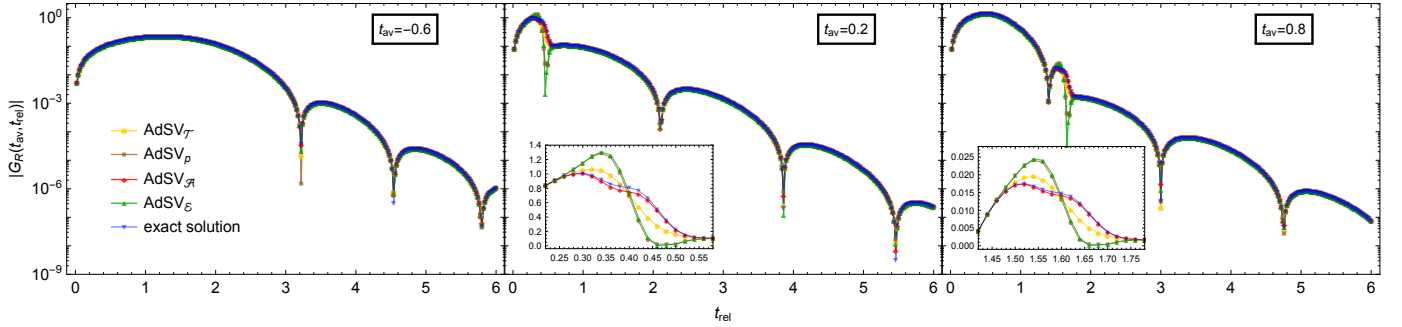


FIG. 2. Comparison of the exact retarded correlation to that obtained from the different AdSV geometries for three values of t_{av} – the insets display zoom-ins on the PP region on a *linear* scale.

in the \mathcal{U} region, we conjecture that the event horizon in the geometry (2) dual to the driven non-equilibrium state can be located by finding that prototype AdSV geometry which approximates to the universal features of $G_R(t_{av}, t_{rel})$ best.

The inset plots in Fig. 2 clearly show that the prototype $AdSV_A$ geometry which reproduces the exact location of the apparent horizon in bulk coordinates (2) provides a remarkably good approximation to $G_R(t_{av}, t_{rel})$ in the PP region where the pump protocol plays a dominant role with about 1 percent relative accuracy. It is remarkable that $AdSV_T$ also provides a reasonably good accuracy in the PP region while $AdSV_E$ and $AdSV_p$ results have large standard deviations from the exact results. We can also check that $AdSV_T$ provides a reasonable approximation to the exact location of the apparent horizon while $AdSV_E$ and $AdSV_p$ fail to do so.

From these results it follows that an effective theory for $G_R(t_{av}, t_{rel})$ including the \mathcal{U} and PP regions (but excluding a tiny POP region) can be obtained simply by knowing how the pump protocol determines the apparent horizon in the dual geometry and then using the simple prototype $AdSV_A$ geometry to reproduce $G_R(t_{av}, t_{rel})$. Our results also provide sufficient support for our conjecture that the exact location of the apparent horizon in the bulk coordinates (2) can be deciphered from $G_R(t_{av}, t_{rel})$ simply by finding which prototype AdSV geometry provides the best approximation in the PP region.

It is interesting to note from Fig. 3 that the prototype $AdSV_A$ geometry gives $\langle t_{00}(t) \rangle$ very accurately. Observe that the curves for $\langle t_{00}(t) \rangle$ of $AdSV_E$ and $AdSV_p$ geometries closely follow each other. For the AdSV geometries, $\langle t_{00}(t) \rangle = 2\langle t_{xx}(t) \rangle$. Since $AdSV_p$ reproduces the numerical pressure $\langle t_{xx}(t) \rangle$ exactly, $AdSV_E$ provides a very good approximation to the pressure. Therefore, the AdSV prototype approximations demonstrate that the event and apparent horizons of the dual geometry carry information about the time-evolution of the pressure and the energy-density respectively.

Finally, as is clear from Fig. 2, the prototype $AdSV_T$ geometry provides a reasonably good approximation for $G_R(t_{av}, t_{rel})$ in the entire $t_{av} - t_{rel}$ plane (except for the tiny POP region) and therefore, someone interested in finding signatures of large- N conformal strong coupling regime in $G_R(t_{av}, t_{rel})$ can readily utilize this $AdSV_T$ geometry constructed from simple measurable inputs.

Concluding remarks: Our results strongly indicate that non-equilibrium behaviour of correlation functions

should play a crucial role in both applications and fundamental understanding of the holographic principle. From the point of view of applications, we have found remarkable universal features of the time-dependent causal correlations that can be reproduced from simple prototype geometric approximations. From the point of view of fundamental understanding, we obtain interesting pointers regarding how we can construct the dual gravitational theory directly from appropriate observables. It is to be noted that some of our results parallel interesting universal behaviour of equal-time correlations and the entanglement entropy during fast quenches [19, 20].

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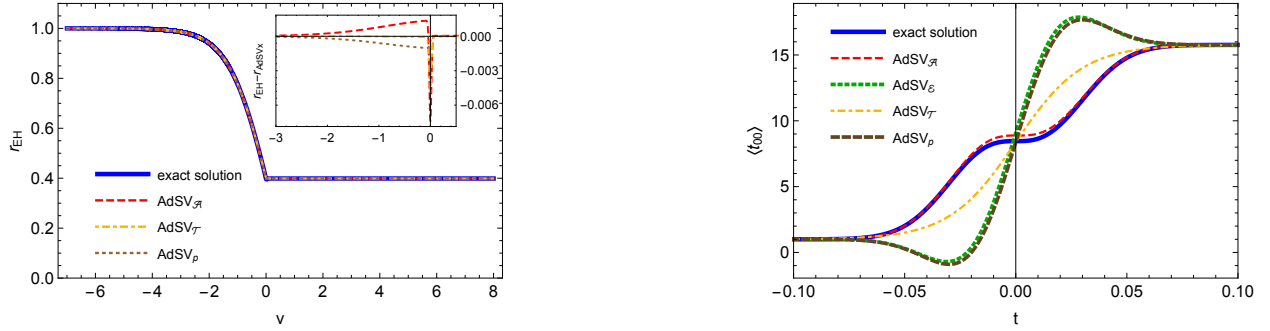


FIG. 3. Left panel: The locations of the event horizon r_{EH} in the different AdSVs and the exact numerical value with the inset focusing on their differences. Right panel: Plots comparing the exact $\langle t_{00}(t) \rangle$ with those obtained from the prototype AdSV geometries.

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Supplemental Material

Here we primarily recapitulate essential methods utilised in this work, which have appeared previously in Refs. [4, 6–9]. Section 4 provides a more detailed account of the construction of the prototype geometries. In the final section we present the quantitative evaluation of the approximations provided by our prototype geometries.

Numerical calculation of the geometry dual to the driven non-equilibrium state using the method of characteristics: The (3 + 1)-D gravitational equations are:

$$R_{MN} + \frac{3}{l^2} G_{MN} = T_{MN}[\Phi] - \frac{1}{2} G_{MN} G^{PQ} T_{PQ}[\Phi],$$

$$(\square - m^2)\Phi = 0,$$

where \square is the Laplacian operator constructed from the metric G_{MN} and

$$T_{MN}[\Phi] = \frac{1}{2} \partial_M \Phi \partial_N \Phi - \frac{1}{4} G_{MN} G^{PQ} \partial_P \Phi \partial_Q \Phi - \frac{1}{2} m^2 \Phi^2.$$

We choose $m^2 = -2/l^2$ as described before. For the sake of convenience, we can set $l = 1$.

Denoting the derivative along the outgoing null direction as $d_+ = \partial_v - A/2 \partial_r$, the equations of motion for the scalar field and the metric (i.e. those of $A(r, v)$, $S(r, v)$ and $\Phi(r, v)$ appearing in (2)) are as follows:

$$\partial_r^2 S + \frac{2}{r} \partial_r S + \frac{(\partial_r \Phi)^2}{4} S = 0, \quad (7a)$$

$$\partial_r d_+ S + \frac{\partial_r S d_+ S}{S} + \frac{S}{4r^2} \left(6 - \frac{m^2}{2} \Phi^2 \right) = 0, \quad (7b)$$

$$r \partial_r^2 \frac{A}{r} + \frac{4 \partial_r S d_+ S}{S^2} - \partial_r \Phi d_+ \Phi = 0, \quad (7c)$$

$$d_+^2 S + \frac{r^2}{2} \partial_r \frac{A}{r^2} d_+ S + \frac{S(d_+ \Phi)^2}{4} = 0, \quad (7d)$$

$$\partial_r(d_+ \Phi) + \frac{\partial_r S d_+ \Phi}{S} + \frac{\partial_r \Phi d_+ S}{S} + \frac{m^2}{2r^2} \Phi = 0. \quad (7e)$$

As described before, the above equations have unique solutions with specified initial and boundary conditions which have been chosen by (i) imposing that in the far past $v_{\text{in}} \ll -3\sigma$ the geometry is a static AdS black brane (planar black hole) with a vanishing Φ field and (ii) the dual system lives in flat Minkowski space and that the energy is pumped through a source $f(v) = \lim_{r \rightarrow 0} \Phi(r, v)/r$ coupling to the operator O dual to the scalar field Φ . The numerical protocol involves the following steps:

1. At initial time $v = v_{\text{in}}$, we solve Eq. (7a) by radial integration to obtain $S(r, v_{\text{in}})$ uniquely using the asymptotic expansion

$$S(r, v) = \frac{1}{r} - \frac{f^2(v)}{8} r + \mathcal{O}(r^2) \quad (8)$$

(obtained from the radial expansion of the equations of motion) which holds for all v and the initial condition $\Phi(r, v_{\text{in}}) = 0$.

2. Next we solve Eq. (7b) to obtain $d_+ S(r, v_{\text{in}})$ by radial integration using the boundary condition $d_+ S(r = 0, v) \approx 1/(2r^2)$ (which holds for all v), the

initial condition $\Phi(r, v_{\text{in}}) = 0$ and the solution for $S(r, v_{\text{in}})$.

3. With the knowledge of $S, d_+ S$ and Φ , we solve Eq. (7e) to obtain $d_+ \Phi(r, v_{\text{in}})$ using the boundary condition $d_+ \Phi(r = 0, v) = -f(v)/2$ (which holds for all v).
4. A can then be found uniquely from Eq. (7c) at $v = v_{\text{in}}$ using the asymptotic expansion

$$A(r, v) = 1 - \frac{f^2(v)}{4} r^2 + a_3(v) r^3 + \mathcal{O}(r^4) \quad (9)$$

(obtained from the radial expansion of the equations of motion) which holds for all v . At $v = v_{\text{in}}$, we need to input $a_3 = -M_{\text{in}}$.

5. From the definition of d_+ it follows that $\partial_v \Phi = d_+ \Phi + (A/2) \partial_r \Phi$ using which we find $\partial_v \Phi$ at initial time v_{in} since A and $d_+ \Phi$ have been obtained in the previous steps. By using a time stepper we then step up to next time $v_{\text{in}} + \Delta v$ to obtain $\Phi(v_{\text{in}} + \Delta v)$.
6. The equation (7d) is actually a constraint, therefore if it is satisfied at $r = 0$ then it should be satisfied for all r . The leading non-trivial asymptotic term of this equation yields the time-evolution of $a_3(v)$ which takes the form:

$$\partial_v a_3 = \frac{1}{2} f(v) (\partial_v^2 f(v) - \partial_v f_1(v)), \quad (10)$$

where $f_1(v) = \lim_{r \rightarrow 0} (1/2) \partial_r^2 \Phi(r, v)$. This equation reproduces the CFT Ward identity corresponding to energy conservation (see below). We can use this to update the value of a_3 to obtain $a_3(v_{\text{in}} + \Delta v)$.

7. Having known $\Phi(r, v_{\text{in}} + \Delta v)$ and $a_3(v_{\text{in}} + \Delta v)$ at the next time step, we start again from step 1 to solve for all the other functions at this time step. We repeat time steps until we reach the final black brane geometry with a vanishing (rather sufficiently small) Φ field.

For the radial integration we have used pseudo-spectral method with 30 grid points and for stepping up in time we have used Adams-Bashforth fourth order time stepper with a time step of $\delta v = .0003$ (which is 0.01σ). A suitable numerical domain $0 < r < r_c$ has been chosen so that the apparent horizon and hence the event horizon of the geometry lies within this domain.

Details of holographic renormalisation: We follow the minimal subtraction scheme to obtain:

$$\langle t_{00} \rangle = \epsilon = \mathcal{C}(-2a_3 + f(\partial_v f - f_1)), \quad (11a)$$

$$\langle t_{xx} \rangle = \langle t_{yy} \rangle = P = -\mathcal{C}a_3, \quad (11b)$$

$$\langle O \rangle = \mathcal{C}(f_1 - \partial_v f), \quad (11c)$$

where $a_3 = \lim_{r \rightarrow 0} \partial_r^3 A(r, v)/6$, $f_1 = \lim_{r \rightarrow 0} \partial_r^2 \Phi(r, v)/2$ and \mathcal{C} is an overall factor proportional to N^2 of the dual theory. We readily see that the above along with Eq. (10) imply the CFT Ward identities

$$\partial_t \langle t^0_0 \rangle = -\partial_t \epsilon = \langle O \rangle \partial_t J, \quad (12a)$$

$$\langle \text{Tr} t \rangle = 2P - \epsilon = (d - \Delta) J \langle O \rangle. \quad (12b)$$

identifying v with the field-theory time coordinate t at the boundary $r = 0$ and J with f as mentioned before, and using $\Delta = 2$ and $d = 3$. In the AdSV geometries, there is no scalar field and hence $\langle O \rangle = 0$, $\langle t_{00} \rangle = -2\mathcal{C}a_3(v)$ and $\langle t_{xx} \rangle = \langle t_{yy} \rangle = -\mathcal{C}a_3(v)$. We normalise the definitions of all one-point functions by dividing them by a suitable factor proportional to N^2 so that we can set $\mathcal{C} = 1$.

Note that for the probe scalar field $\tilde{\Phi}$, the same formula Eq. (11c) applies to the dual $\langle \tilde{O} \rangle$ operator both in the numerical and the AdSV geometries with f and f_1 replaced by $\tilde{f} = \lim_{r \rightarrow 0} \tilde{\Phi}(r, v)/r$ and $\tilde{f}_1 = \lim_{r \rightarrow 0} \tilde{\Phi}(r, v)/r^2$ respectively since $\langle \tilde{O} \rangle$ has the same scaling dimension $\Delta = 2$. It is easy to see that the contact term $-\partial_v \tilde{f} = -\partial_t \tilde{J}$ in $\langle \tilde{O} \rangle$ gives a $-\partial_{t_{\text{rel}}} \delta(t_{\text{rel}})$ contribution to $G_R(t_{\text{av}}, t_{\text{rel}})$. After doing Wigner transform, i.e. the Fourier transform of t_{rel} dependence (defined according to our sign convention), we obtain

$$G_R(t_{\text{av}}, \omega) = \int_{-\infty}^{\infty} e^{-i\omega t_{\text{rel}}} G_R(t_{\text{av}}, t_{\text{rel}}) dt_{\text{rel}}. \quad (13)$$

The contact term $-\partial_{t_{\text{rel}}} \delta(t_{\text{rel}})$ then produces $-i\omega$ term in $G_R(t_{\text{av}}, \omega)$. This implies that the spectral function

$$\rho(t_{\text{av}}, \omega) = -2\text{Im}G_R(t_{\text{av}}, \omega) \quad (14)$$

gets the state-independent contribution 2ω and we have checked that this term ensures that $\rho(\omega, t_{\text{av}}) > 0$ for $\omega > 0$ as should follow from the spectral representation in the dual field theory.

Numerical calculation of the non-equilibrium retarded correlation function: The Klein-Gordon equation for the scalar field $\tilde{\Phi}(r, v, \mathbf{k})$ dual to the operator \tilde{O} whose correlation function we are calculating takes the same form as (7e) for $\mathbf{k} = 0$ with $S(r, v)$ and $A(r, v)$ fixed by their forms in the background geometry dual to the driven non-equilibrium state. The method of characteristics can then be readily implemented provided we specify $\tilde{\Phi}(r, v)$ at initial time and the source $\tilde{f}(v) = \lim_{r \rightarrow 0} \tilde{\Phi}(r, v)/r$ for all v as should be clear from our prior discussion. In the AdSV backgrounds, $S(r, v) = 1/r$ and $A(r, v) = 1 - M(v)r^3$. In order to obtain $G_R^{\tilde{O}\tilde{O}}(t, t')$, we need to set initial conditions $\tilde{\Phi}(r, v_{\text{in}}) = 0$ and $\tilde{f}(v) = \delta(v - t')$ as discussed in the main text. The choice $\tilde{f}(v) = \delta(v - t')$ and (11c) simply imply that $G_R^{\tilde{O}\tilde{O}}(t, t') = \tilde{f}_1(t) - \partial_t \delta(t - t')$.

We now show that numerically the delta function limit for the source $\tilde{f}(v)$ can be taken by assuming it to be a normalised Gaussian function

$$\tilde{f}(v) = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} e^{-\frac{(v-t')^2}{2\tilde{\sigma}^2}}, \quad (15)$$

with root-mean-square-width $\tilde{\sigma}$ (not to be confused with the σ of the pump $f(t)$) and taking the limit $\tilde{\sigma} \rightarrow 0$. We have plotted $|\tilde{f}_1(t)|$ in Fig. 4 for various values of $\tilde{\sigma}$. We readily see that the behaviour of $\tilde{f}_1(t)$ converges with decreasing $\tilde{\sigma}$ for all times t up to any desired order to numerical accuracy. For practical purposes, we choose to work with the width $\tilde{\sigma} = .01$ which allows to take the Dirac delta limit at the level of fourth digit precision.

Construction of the prototype geometries $\text{AdSV}_{\mathcal{A}}$ and $\text{AdSV}_{\mathcal{E}}$: We first address the issue regarding how can we compare the numerical geometry (2) representing the dual of the driven non-equilibrium state with the pro-

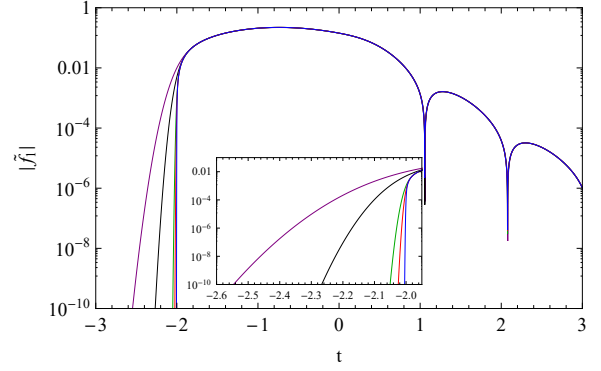


FIG. 4. For the Gaussian source \tilde{f} centered at $t' = -2$, $|\tilde{f}_1|$ is plotted on a log scale for various values of $\tilde{\sigma}$. The purple, black, green, red and blue colours correspond to the widths $\tilde{\sigma} = 0.1, 0.05, 0.01, 0.005$, and 0.001 respectively.

totype AdSV geometries (5) representing simple density matrices given that they are different spacetimes. Firstly, of course for all these geometries we are choosing the ingoing Eddington-Finkelstein gauge where

$$G_{rr} = G_{rx} = G_{ry} = 0, \quad G_{rv} = -l^2/r^2. \quad (16)$$

Secondly, all of these geometries coincide at very early and very late times corresponding to AdS black branes (planar black holes) with masses M_{in} and M_f respectively. This however does not completely fix the choice of coordinates in these geometries exactly because the gauge (16) has residual diffeomorphism symmetries corresponding to

$$r \rightarrow r + \lambda(v), \quad v \rightarrow v + v_0 \quad (17)$$

with v_0 being a constant. Note that actually if we replace $\lambda(v)$ in the radial diffeomorphism above by $\lambda(v, x, y)$, the gauge (16) is still preserved but the metric is no longer homogeneous manifestly. Also note that under this (radial) diffeomorphism, the functions A and S transform to \tilde{A} and \tilde{S} respectively which have different asymptotic behaviours at $r = 0$. Therefore this diffeomorphism symmetry is simply fixed by our boundary conditions $\lim_{r \rightarrow 0} A = \lim_{r \rightarrow 0} S = 1$. The second residual diffeomorphism symmetry corresponding to time translation is fixed in the numerical geometry (2) by choosing the time in which the pump $f(t)$ peaks to be $t = 0$ (recall that at $r = 0$, the bulk time-coordinate v coincides with the field-theory time-coordinate t). In the prototype AdSV geometries, this time translation freedom is fixed by a suitable form of matching with the numerical geometry. In $\text{AdSV}_{\mathcal{T}}$ where $M(v)$ is chosen to be as shown in (6), the origin of time is chosen by demanding that the midpoint of the tanh function coincides with the time when the pumping is peaked at the boundary. In AdSV_p , this is fixed by construction via the exact matching of $a_3(v)$ (i.e. $P(t)$) with that obtained from the numerical geometry. Similarly, it will be fixed in the $\text{AdSV}_{\mathcal{A}}$ and $\text{AdSV}_{\mathcal{E}}$ by construction via the exact matching of the location of the apparent and the event horizons as described below.

The apparent horizon $r_{\text{AH}}^{\text{exact}}(v)$ of the numerical geometry (2) can be found by solving

$$d_+ S(v, r_{\text{AH}}^{\text{exact}}(v)) = 0 \quad (18)$$

since $r = r_{\text{AH}}^{\text{exact}}(v)$ is a surface of vanishing extrinsic curvature. In the prototype AdSV geometries (5), this equation

simplifies because $S(r, v) = 1$ and $A(r, v)$ takes a simple form determined by the choice of the mass function $M(v)$ so that the location of the apparent horizons are given simply by $r_{\text{AH}}^{\text{AdSV}^3}(v) = M(v)$. Therefore, in order to construct a prototype $\text{AdSV}_{\mathcal{A}}$ geometry which can reproduce the location of the apparent horizon of the actual numerical background $r_{\text{AH}}^{\text{exact}}(v)$, we need to choose the black hole mass function $M(v)$ as:

$$M(v) = \frac{1}{r_{\text{AH}}^{\text{exact}^3}(v)}. \quad (19)$$

This completes the construction of $\text{AdSV}_{\mathcal{A}}$.

The location of the event horizon $r_{\text{EH}}(v)$ is given by the null geodesic which coincides with the location of the final black hole horizon in the limit $v \rightarrow \infty$ i.e. in the far future. For the numerical geometry (2), the location of the event horizon can be found by solving the differential equation

$$\partial_v r_{\text{EH}}^{\text{exact}}(v) + A(v, r_{\text{EH}}^{\text{exact}}(v))/2 = 0. \quad (20)$$

subject to the future boundary condition of equilibrium horizon, i.e., $r_{\text{EH}}^{\text{exact}}(v \rightarrow \infty) = M_{\text{f}}^{1/3}$. In the AdSV prototype geometries (5), the location of the event horizon $r_{\text{EH}}^{\text{AdSV}}(v)$ is given by the same equation but with $A(r, v)$ taking the simpler form determined by the choice of the mass function $M(v)$. It follows that in order to construct a prototype geometry $\text{AdSV}_{\mathcal{E}}$ whose event horizon coincides with that of the actual numerical geometry, the mass function $M(v)$ should be chosen to satisfy

$$M(v) = \frac{1}{r_{\text{EH}}^{\text{exact}^3}(v)} (1 + 2\partial_v r_{\text{EH}}^{\text{exact}}(v)). \quad (21)$$

This completes the construction of $\text{AdSV}_{\mathcal{E}}$.

Quantitative comparison of $G_R(t_{\text{av}}, t_{\text{rel}})$ obtained in the prototype geometries: In order to quantify how well the prototype geometries approximate the exact $G_R(t_{\text{av}}, t_{\text{rel}})$ we need to compute

$$\Sigma_x = \frac{\int_{\mathcal{R}} dt_{\text{av}} dt_{\text{rel}} \frac{|G_R(t_{\text{av}}, t_{\text{rel}}) - G_R^{\text{AdSV}_x}(t_{\text{av}}, t_{\text{rel}})|}{|G_R(t_{\text{av}}, t_{\text{rel}})|}}{\int_{\mathcal{R}} dt_{\text{av}} dt_{\text{rel}}}, \quad (22)$$

where x is \mathcal{E} , \mathcal{A} , \mathcal{T} or p , and \mathcal{R} denotes either the universal region \mathcal{U} or the probe on pump (PP) region. This we approximate by a Riemann sum over the plaquettes $\Delta t_{\text{av}} \Delta t_{\text{rel}}$. Furthermore, we introduce a cutoff by demanding $|G_R(t_{\text{av}}, t_{\text{rel}})| > 10^{-4}$ (in units $T_{\text{in}} = 1$) in both integrands of (22) in order to tame numerical errors.

In the region \mathcal{U} we find that all of the prototype geometries deviate from the exact result by less than 1 percent on average. In particular $\text{AdSV}_{\mathcal{E}}$ and $\text{AdSV}_{\mathcal{T}}$ perform equally well (according to our expectation because $\text{AdSV}_{\mathcal{T}}$ approximates the location of the event horizon very well) although $\text{AdSV}_{\mathcal{E}}$ gives a slightly better approximation. These are followed by $\text{AdSV}_{\mathcal{A}}$ while AdSV_p performs the worst. In the PP region again AdSV_p is the worst approximation while the ranking of the others is exactly reversed – both $\text{AdSV}_{\mathcal{A}}$ followed by $\text{AdSV}_{\mathcal{T}}$ are better than 1 percent while both $\text{AdSV}_{\mathcal{E}}$ followed by AdSV_p deviate by a bit more than 1 percent on average. We have also checked the POP region, where both the observation time as well as the probe time reside within the pump duration. However, none of the prototype geometries give reasonable results there (with the same ranking as in the PP region but deviations are up to 50 percent on average).